Problem 6

For the following series, write formulas for the sequences a_n, S_n, and R_n, and find the limits of the sequences as $n \to \infty$ (if the limits exist).

$$\sum_{1}^{\infty} \frac{1}{n(n+1)} \qquad \textit{Hint:} \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Solution

$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$S_n = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$= \frac{n}{n+1}$$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$$

$$R_n = S - S_n = 1 - \left(1 - \frac{1}{n+1}\right) = \frac{1}{n+1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n(n+1)} = 0$$

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{1}{n+1} = 0$$